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Recursivity and geometry of the hypercube

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Abstract

It is a difficult theoretical and computational problem to describe explicitly the list of hyperplanes spanned by the vertices of the n-cube.

In this paper we describe a procedure to generate hyperplanes of the n-cube from hyperplanes of the $(n-1)$-cube. Our main theorem says that starting with the hyperplanes of the 1-cube and iterating this procedure we obtain the complete list of hyperplanes of the n-cube up till $n = 6$. For $n = 7$, with this procedure, not all the hyperplanes are obtained. An explicit description of the hyperplanes of the 7-*cube* is given, followed by a brief analysis of some further consequences of the results obtained.

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1. Introduction

Many easily stated questions regarding the geometry of the n-cube are still unsettled and known as computationally difficult problems. As examples we have the well known problem of finding a largest j-simplex with vertices in the n-cube (see [5]

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for a survey) or determining the number of hyperplanes spanned by the vertices of n-cube (see [1,7]). This is essentially the problem we consider in this paper: denumerating and encoding the list of affine hyperplanes spanned by the vertices of the n-cube.

This problem is a particular case of the problem of describing explicitly the n-cube matroid (see [3,6,2]).

An experimental exhaustive enumeration of the hyperplanes of the n-cube up till dimension 8 has been carried out by Aischolzer and Aurenhammer [1] and the authors estimated in 35 years the running time needed (with their program and machine) to compute the answer for the next dimension, 9. As Aischolzer and. Aurenhammer report in [1], the number of hyperplanes of the n-cube seems to grow superexponentially with n and "in each further dimension we encountered new kinds of hyperplanes which could not be brought into connection with hyperplanes of lower dimensions".

In this paper we make a first attempt towards a recursive procedure to determine the hyperplanes of the n-cube.

We identify the family of hyperplanes of the n-cube with a family $\mathscr{H}_n$ of non-negative integer vectors of $\mathbb{R}^{n+1}$. Proposition 2.6, describes a natural procedure to generate hyperplanes of the n-cube from hyperplanes of the $(n-1)$-cube, i.e. vectors of $\mathscr{H}_n$ from vectors of $\mathscr{H}_{n-1}$. We introduce then (Definition 2.4) the sequence $\mathscr{G}_n \subseteq \mathscr{H}_n$ which represents the family of hyperplanes of the n-cube generated recursively from the family of hyperplanes of the 1-cube applying successively the procedure defined in Proposition 2.6. Our main theorem asserts that the sequence $\mathscr{G}_n$ defines all the hyperplanes of the n-cube for $n \leqslant 6$, i.e. $\mathscr{G}_n = \mathscr{H}_n$, $\forall n \leqslant 6$ but for $n = 7$ $\mathscr{G}_7$ is strictly contained in $\mathscr{H}_7$. An explicit description of the set $\mathscr{H}_7$ in terms of $\mathscr{G}_7$ is also given in Theorem 2.1. Some further consequences of the results obtained are briefly presented in the final remarks.

2. The families $\tilde{\mathscr{H}}_n$, $\mathscr{H}_n$ and $\mathscr{G}_n$

Notation

We fix as *n-cube* the set $C^n = \{-1, 1\}^n$, the set of vertices of the cube $[-1, 1]^n$ of $\mathbb{R}^n$.

We consider $\mathbb{R}^n$ either as the real linear or affine space. The canonical inner product of two vectors of $\mathbb{R}^n$ is denoted $\mathbf{x}|\mathbf{y}$.

Given a subset $A \subseteq \mathbb{R}^n$, the affine subspace generated by A is denoted $\mathrm{aff}(A)$.

A hyperplane of $\mathbb{R}^n$ is an affine subspace of $\mathbb{R}^n$ of dimension $n-1$ or, equivalently, the set of solutions of a linear equation. Given $\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $b \in \mathbb{R}$ we denote by $H : \mathbf{x}|\mathbf{h} = b$ the hyperplane of $\mathbb{R}^n$ defined by the equation $\sum_{i=1}^{n} h_i x_i = b$.

Given a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\forall i = 1, \ldots, n$ we denote by $\mathbf{x}(\hat{i})$ the vector $\mathbf{x}(\hat{i}) := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$. A vector $\mathbf{x} = (x_1, \ldots, x_n)$ will be denoted $\mathbf{x} = (\mathbf{x}(\hat{n}), x_n)$ or $\mathbf{x} = (x_1, \mathbf{x}(\hat{1}))$.

Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we will write, $\forall i \in \{1, \ldots, n\}$ $\mathbf{x}|\mathbf{y} = \mathbf{x}(\hat{i})|\mathbf{y}(\hat{i}) + x_i y_i$.

Definition 2.1. A hyperplane of C^n is an affine hyperplane of $\mathbb{R}^n$ spanned by vertices of C^n. Given $\mathbf{h} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $b \in \mathbb{R}$ we say that the hyperplane $H : \mathbf{x}|\mathbf{h} = b$ is a hyperplane of C^n if $H = \text{aff}(H \cap C^n)$.

Example 1. The hyperplanes of the 2-cube $C^2 := \{-1, 1\}^2$ are the 6 hyperplanes of $\mathbb{R}^2$ defined by:

$$
\begin{array}{lll}
H_1 : (x, y)|(1, 0) = 1 & \text{or} & H_1 = \text{aff}(\{(1, 1), (1, -1)\}),\\
H_2 : (x, y)|(1, 0) = -1 & \text{or} & H_2 = \text{aff}(\{(-1, 1), (-1, -1)\}),\\
H_3 : (x, y)|(0, 1) = 1 & \text{or} & H_3 = \text{aff}(\{(1, 1), (-1, 1)\}),\\
H_4 : (x, y)|(0, 1) = -1 & \text{or} & H_4 = \text{aff}(\{(1, -1), (-1, -1)\}),\\
H_5 : (x, y)|(1, 1) = 0 & \text{or} & H_5 = \text{aff}(\{(-1, 1), (1, -1)\}),\\
H_6 : (x, y)|(1, -1) = 0 & \text{or} & H_6 = \text{aff}(\{(1, 1), (-1, -1)\}).
\end{array}
$$

Definition 2.2 (*The family* $\tilde{\mathscr{H}}_n$). The set $\tilde{\mathscr{H}}_n$ is the subset of $\mathbb{R}^{n+1}$ defined by: $\tilde{\mathscr{H}}_n := \left\{\mathbf{h} = (\mathbf{h}(\widehat{n+1}), h_{n+1}) \in \mathbb{R}^{n+1} : \quad H : \mathbf{x}|\mathbf{h}(\widehat{n+1}) = h_{n+1} \text{ is a hyperplane of } C^n\right\}$.

Given a vector $\mathbf{h} \in \tilde{\mathscr{H}}_n$ define the class of $\mathbf{h}$ as the set $[\mathbf{h}] := \{\alpha\mathbf{h}, \alpha \in \mathbb{R} \setminus \{0\}\}$. It is clear that we can identify each hyperplane $H : \mathbf{x}|\mathbf{h} = b$ of C^n with the class of the vector $(\mathbf{h}, b)$. The next proposition states that for every such hyperplane we can choose a representative of such class with integer coordinates. The proof is elementary linear algebra and is omitted.

Proposition 2.1. *Consider* $(\mathbf{h}, b) \in \mathbb{R}^n \times \mathbb{R}$ *such that the hyperplane* $H : \mathbf{x}|\mathbf{h} = b$ *is a hyperplane of* C^n. *Then, there exists* $\alpha \in \mathbb{R}$ *such that* $(\alpha\mathbf{h}, \alpha b) \in \mathbf{Z}^n \times \mathbf{Z}$.

Remark 2.1. The decision problem: Given $(\mathbf{h}(\widehat{n+1}), h_{n+1}) \in \mathbf{Z}^{n+1}$ decide if $(\mathbf{h}(\widehat{n+1}), h_{n+1}) \in \mathscr{H}_n$ comprises two steps:

(1) Determining the set $V := \{\mathbf{v} \in C^n : \mathbf{v}|\mathbf{h}(\widehat{n+1}) = h_{n+1}\}$, the set of vertices of C^n in the hyperplane $H : \mathbf{v}|\mathbf{h}(\widehat{n+1}) = h_{n+1}$.
(2) Verifying that $H = \text{aff}(V)$.

Since the decision problem: Given $(\mathbf{h}(\widehat{n+1}), h_{n+1}) \in \mathbf{Z}^{n+1}$ decide if there exists $\mathbf{v} \in C^n$ such that $\mathbf{v}|\mathbf{h}(\widehat{n+1}) = h_{n+1}$ is NP-complete [4], it is clear that the above problem is also NP-complete.

The next two propositions allow us to study the family of hyperplanes of C^n by studying a subset of vectors of $\mathbb{N}_0^{n+1}$, the family $\mathscr{H}_n$ (see Definition 2.3). The relation between the two families, $\mathscr{H}_n$ and $\tilde{\mathscr{H}}_n$, is made explicit in Proposition 2.4.

Proposition 2.2. *Consider a vector* $(\mathbf{h}(\widehat{n+1}), h_{n+1}) = (h_1, \ldots, h_n, h_{n+1}) \in \tilde{\mathscr{H}}_n$. *Then*:

1. *For every permutation* $\sigma \in S_n$, $(\mathbf{h}(\widehat{n+1})_\sigma, h_{n+1}) \in \tilde{\mathscr{H}}_n$, *where* $\mathbf{h}(\widehat{n+1})_\sigma := (h_{\sigma(1)}, \ldots, h_{\sigma(n)})$.
2. $(\mathbf{h}(\widehat{n+1})^+, h_{n+1}) \in \tilde{\mathscr{H}}_n$, *where* $\mathbf{h}(\widehat{n+1})^+ = (|h_1|, \ldots, |h_n|)$.
3. $(\mathbf{h}(\widehat{n+1}), -h_{n+1}) \in \tilde{\mathscr{H}}_n$.

Proof. (1) is an immediate consequence of the fact that for every permutation $\sigma \in S_n$ the linear map $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ defined, in relation to the canonical basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ of $\mathbb{R}^n$, by $f_\sigma(\mathbf{e}_i) = \mathbf{e}_{\sigma(i)}$ is a symmetry of C^n which maps the hyperplane $H : \mathbf{x}|\mathbf{h}(\widehat{n+1}) = h_{n+1}$ onto the hyperplane $f_\sigma(H) : \mathbf{x}|\mathbf{h}(\widehat{n+1})_\sigma = h_{n+1}$.

(2) For every $i = 1, \ldots, n$, the reflexion $R_i : \mathbb{R}^n \to \mathbb{R}^n$ on the hyperplane $H_i : x_i = 0$ of $\mathbb{R}^n$ is a symmetry of C^n. Therefore, setting $I := \{i \in \{1, \ldots, n\} : h_i < 0\}$, the composition $f = \circ_{i \in I} R_i$ is a symmetry of C^n which maps the hyperplane $H : \mathbf{x}|\mathbf{h}(\widehat{n+1}) = h_{n+1}$ onto the hyperplane $f(H) : \mathbf{x}|\mathbf{h}(\widehat{n+1})^+ = h_{n+1}$.

(3) The inversion $f : \mathbb{R}^n \to \mathbb{R}^n$ with center the origin $\mathbf{0} \in \mathbb{R}^{\mathbf{n}}$, defined by $f(\mathbf{x}) = -\mathbf{x}$, is a symmetry of C^n which maps the hyperplane $H : \mathbf{x}|\mathbf{h}(\widehat{n+1}) = h_{n+1}$ onto the hyperplane $f(H) : \mathbf{x}|\mathbf{h}(\widehat{n+1}) = -h_{n+1}$. $\square$

Proposition 2.3. *For a vector* $\mathbf{h} = (h_1, \ldots, h_{n+1}) \in \mathbb{R}^{\mathbf{n+1}}$ *the following three conditions are equivalent*:

1. $\mathbf{h}$ *is a vector of* $\tilde{\mathscr{H}}_n$.
2. *The hyperplane* H *of* $\mathbb{R}^{n+1}$ *defined by* $H : \mathbf{x}|\mathbf{h} = 0$ *is a hyperplane of* C^{n+1}.
3. $\forall i \in \{1, \ldots, n+1\}$ *the vector* $\mathbf{h}(i) = (\mathbf{h}(\hat{i}), h_i)$ $(:= (h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_{n+1}, h_i))$ *is a vector of* $\tilde{\mathscr{H}}_n$.

Proof. To prove (1) $\Rightarrow$ (2), assume that $H_{n+1} : \mathbf{x}|\mathbf{h}(\widehat{n+1}) = h_{n+1}$ is a hyperplane of C^n. Let H be the hyperplane of $\mathbb{R}^{n+1}$ defined by $H : \mathbf{x}|(\mathbf{h}((\widehat{n+1})), h_{n+1}) = 0$. We have to prove that $H = \mathrm{aff}(H \cap C^{n+1})$.

First observe that a point $\mathbf{v} = (\mathbf{v}(\widehat{n+1}), v_{n+1})$ belongs to $H \cap C^{n+1}$ if and only if $(\mathbf{v}(\widehat{n+1}), v_{n+1})|(\mathbf{h}(\widehat{n+1}), h_{n+1}) = 0 \iff \mathbf{v}(\widehat{n+1})|\mathbf{h}(\widehat{n+1}) = -v_{n+1}h_{n+1}$. If $v_{n+1} = 1$ then $-\mathbf{v}(\widehat{n+1}) \in H_{n+1} \cap C^n$. If $v_{n+1} = -1$ then $\mathbf{v}(\widehat{n+1}) \in H_{n+1} \cap C^n$.

Define $H^+ := \{(-\mathbf{v}, 1) \in C^{n+1} : \mathbf{v} \in H_{n+1} \cap C^n\}$ and $H^- := \{(\mathbf{v}, -1) \in C^{n+1} : \mathbf{v} \in H_{n+1} \cap C^n\}$, then $H \cap C^{n+1} = H^+ \cup H^-$. Now we have to prove that $\dim(\mathrm{aff}(H^+ \cup H^-)) = n$.

By definition of H^+ and H^- it is clear, since H_{n+1} is a hyperplane of C^n, that $\dim(\mathrm{aff}(H^+)) = \dim(\mathrm{aff}(H^-)) = \dim(\mathrm{aff}(H_{n+1} \cap C^n)) = n - 1$. It is also clear that $H^- \cap \mathrm{aff}(H^+) = \emptyset$ and therefore $\dim(\mathrm{aff}(H^+ \cup H^-)) > n - 1$. Since $H^+ \cup H^-$ is contained in the hyperplane H of $\mathbb{R}^{n+1}$ we conclude that $\dim(\mathrm{aff}(H^+ \cup H^-)) = \dim(H) = n$ and $H = \mathrm{aff}(H \cap C^{n+1})$.

Proof of (2) $\Rightarrow$ (3). Assuming that $H : \mathbf{x}|\mathbf{h} = 0$ is a hyperplane of C^{n+1} we now prove that $\forall i \in \{1, \ldots, n+1\}$ the hyperplane H_i of $\mathbb{R}^n$ defined by $H_i : \mathbf{x}|\mathbf{h}(\hat{i}) = h_i$ is a hyperplane of C^n:

Consider $i \in \{1, \ldots, n+1\}$ and let H_i^+, H_i^- be the subsets of C^{n+1} defined by:

$$H_i^+ := \left\{\mathbf{v} = (v_1, \ldots, v_{n+1}) \in H \cap C^{n+1} : \ \ v_i = 1\right\}$$

and

$$H_i^- := \{\mathbf{v} = (v_1, \ldots, v_{n+1}) \in H \cap C^{n+1} : \ \ v_i = -1\}.$$

Since H is a hyperplane of C^{n+1} we have $\mathrm{aff}(H_i^+ \cup H_i^-) = H$. On the other hand H is a hyperplane of $\mathbb{R}^{n+1}$ that contains the origin $\mathbf{0}$ implying that: $\mathbf{v} \in H_i^+ \iff -\mathbf{v} \in H_i^-$. Since the origin $\mathbf{0}$ is not in C^{n+1}, every point $\mathbf{v} \in H_i^+$ is in the affine line that contains $\mathbf{0}$ and $-\mathbf{v}$ and therefore $H = \mathrm{aff}(H_i^+ \cup H_i^-) = \mathrm{aff}(H_i^- \cup \{\mathbf{0}\})$. Consequently $\dim(\mathrm{aff}(H_i^-)) = n - 1$.

Let $V_i := \{\mathbf{w} = (w_1, \ldots, w_n) \in C^n : (w_1, \ldots, w_{i-1}, -1, w_i, \ldots, w_n) \in H_i^-\}$. It is clear that $\dim(\mathrm{aff}(V_i)) = \dim(\mathrm{aff}(H_i^-)) = n - 1$, moreover, by definition of H_i^- we have:

$$(w_1, \ldots, w_{i-1}, -1, w_i, \ldots, w_n)|(h_1, \ldots, h_{n+1}) = 0 \iff \mathbf{w}|\mathbf{h}(\hat{\imath}) = h_i.$$

Therefore H_i is an hyperplane of C^n.

It is clear that (3) $\Rightarrow$ (1). $\square$

Corollary 2.1. *For a vector* $\mathbf{h} = (h_1, \ldots, h_{n+1}) \in \mathbb{R}^{\mathbf{n+1}}$ *the following three conditions are equivalent*:

1. *There exists* $i \in \{1, \ldots, n+1\}$ *such that the hyperplane* $H_i : \mathbf{x}|\mathbf{h}(\hat{\imath}) = h_i$ *is a hyperplane of* C^n.
2. $\forall i \in \{1, \ldots, n+1\}$ *the hyperplane* $H_i : \mathbf{x}|\mathbf{h}(\hat{i}) = h_i$ *is a hyperplane of* C^n.
3. *The hyperplane* H *of* $\mathbb{R}^{n+1}$ *defined by* $H : \mathbf{x}|\mathbf{h} = 0$ *is a hyperplane of* C^{n+1}.

Proof. (1) $\Rightarrow$ (3) is an adaptation, for $i \neq n+1$, of the proof of the implication (1) $\Rightarrow$ (2) in Proposition 2.3. The proof that (3) $\Rightarrow$ (2) was done in Proposition 2.3 (2) $\Rightarrow$ (1) is obvious. $\square$

We can now define the family $\mathscr{H}_n$ of nonnegative integer vectors of $\mathbb{R}^{n+1}$ that we will use to encode the hyperplanes of C^n.

Definition 2.3 (*The family* $\mathscr{H}_n$). $\mathscr{H}_n := \{\mathbf{h} = (h_1, \ldots, h_{n+1}) \in \mathbb{N}_0^{\mathbf{n+1}} : h_1 \leqslant \ldots \leqslant h_{n+1}, \gcd(h_1, \ldots, h_{n+1}) = 1$ and $\mathbf{h}$ satisfies one of the conditions of Corollary 2.1$\}$.

The next proposition is an immediate consequence of Propositions 2.2 and 2.3 and describes the relation between the families $\mathscr{H}_n$ and $\tilde{\mathscr{H}}_n$.

Proposition 2.4 (*Relation between $\tilde{\mathscr{H}}_n$ and $\mathscr{H}_n$*).

1. *For every $\mathbf{g} = (g_1, \ldots, g_{n+1}) \in \tilde{\mathscr{H}}_n$ and every permutation $\sigma \in S_{n+1}$ such that $|g_{\sigma(1)}| \leqslant \cdots \leqslant |g_{\sigma(n+1)}|$ there exists a unique $\alpha \in \mathbb{R}$ such that the vector $\alpha \mathbf{g}_\sigma := (\alpha|g_{\sigma(1)}|, \ldots, \alpha|g_{\sigma(n+1)}|)$ is a vector of $\mathscr{H}_n$. The vector $\alpha\mathbf{g}_\sigma$ is the vector of $\mathscr{H}_n$ defined by $\mathbf{g}$.*
2. $\tilde{\mathscr{H}}_n = \{\alpha \mathbf{h}_{\sigma,\epsilon} = \alpha(\epsilon_1 h_{\sigma(1)}, \ldots, \epsilon_{n+1} h_{\sigma(n+1)}) \in \mathbb{R}^{\mathbf{n+1}} : \quad \mathbf{h} = (h_1, \ldots, h_{n+1}) \in \mathscr{H}_n, \sigma \in S_{n+1}, \epsilon_1, \ldots, \epsilon_{n+1} \in \{-1, 1\}, \alpha \in \mathbb{R} \setminus \{0\}\}$.

Remark 2.2. Note that each vector of $\mathscr{H}_n$ is of the form:

$$(\underbrace{0, \ldots, 0}_{p_0}, \underbrace{n_1, \ldots n_1}_{p_1}, \ldots, \underbrace{n_k, \ldots n_k}_{p_k}),$$

with $0 < n_1 < \cdots < n_k$, $p_0 + p_1 + \cdots + p_k = n + 1$, and determines exactly $2^{(n-p_0)} \times \frac{(n+1)!}{p_0! p_1! \cdots p_k!}$ distinct hyperplanes of C^n.

A direct calculation of the family of hyperplanes of C^n for $n = 1, 2, 3$ leads the first three terms of the sequence $\{\mathscr{H}_n\}_{n\in\mathbb{N}}$:

Proposition 2.5. *The family $\mathscr{H}_n$ for $n = 1, 2, 3$ is*:

$\mathscr{H}_1 = \{(1, 1)\}$ *which encodes the two hyperplanes of the* 1*-cube.*
$\mathscr{H}_2 = \{(0, 1, 1)\}$ *and encodes the six hyperplanes of the* 2*-cube.*
$\mathscr{H}_3 = \{(0, 0, 1, 1), (1, 1, 1, 1)\}$ *and encodes the* 20 *hyperplanes of the* 3*-cube.*

The geometrical idea behind the next proposition is the following: identify in the natural way C^n with $C^n \times \{1\}$ of $\mathbb{R}^{n+1}$. Every hyperplane H of the cube C^n is then identified with the hyperplane $H \times \{1\}$ of the cube $C^n \times \{1\}$. If $\mathbf{v_1}, \ldots, \mathbf{v_n} \in C^n$ are such that $(\mathbf{v}_1, 1), \ldots, (\mathbf{v}_n, 1)$ is an affine basis of the hyperplane $H \times \{1\}$ then for every $\mathbf{v} \in C^n$, $\mathrm{aff}(\{(\mathbf{v_1}, 1), \ldots, (\mathbf{v_n}, 1), (\mathbf{v}, -1)\}$ is a hyperplane of C^{n+1}. Note that this idea is a generalization of the implication $1 \Rightarrow 2$) of Proposition 2.3.

Proposition 2.6. *Consider a vector $\mathbf{h} = (h_1, \ldots, h_{n+1}) \in \mathscr{H}_n$. Then,*

1. *For every $i \in \{1, \ldots, n+1\}$ and every $\mathbf{v} \in C^n$, the vector $\mathbf{G}_{\mathbf{h}(\hat{i}),\mathbf{v}}$ defined by*:

$$\mathbf{G}_{\mathbf{h}(\hat{i}),\mathbf{v}} := \left(\mathbf{h}(\hat{i}), \frac{\mathbf{h}(\hat{i})|\mathbf{v} - h_i}{2}, \frac{\mathbf{h}(\hat{i})|\mathbf{v} + h_i}{2}\right)$$

is a vector of $\tilde{\mathscr{H}}_{n+1}$ and defines a (unique) vector of $\mathscr{H}_{n+1}$ which will be denoted $\mathbf{G}(\mathbf{h}, i, \mathbf{v})$.

2. *If* $\mathbf{h} \in \mathscr{H}_n$ *and* $i \in \{1, \ldots, n+1\}$ *then for all pairs of opposite vectors* $\mathbf{v}, -\mathbf{v} \in C^n$ *the vectors* $\mathbf{G}_{\mathbf{h}(\hat{i}),\mathbf{v}}$ *and* $\mathbf{G}_{\mathbf{h}(\hat{i}),-\mathbf{v}}$ *define the same vector* $\mathbf{G}(\mathbf{h}, i, \mathbf{v}) \in \mathscr{H}_{n+1}$.

Proof. To prove (1) consider $\mathbf{h} \in \mathscr{H}_n$, $i \in \{1, \ldots, n+1\}$ and $\mathbf{v} \in C^n$. Let H be the subset of C^{n+1} defined by:

$$H := \left\{(\mathbf{w}, 1) \in C^{n+1} : \mathbf{w} \in C^n \text{ and } \mathbf{w}|\mathbf{h}(\hat{i}) = h_i\right\}$$
$$\cup \left\{(\mathbf{w}, -1) \in C^{n+1} : \mathbf{w} \in C^n \text{ and } \mathbf{w}|\mathbf{h}(\hat{i}) = \mathbf{v}|\mathbf{h}(\hat{i})\right\}.$$

The set H has the following two properties, whose proof we leave to the reader:

(i) Every element of H belongs to the hyperplane $H_{i,\mathbf{v}}$ of $\mathbb{R}^{n+1}$ defined by $H_{i,\mathbf{v}}$: $\mathbf{x}|\left(\mathbf{h}(\hat{i}), \frac{\mathbf{h}(\hat{i})|\mathbf{v}-h_i}{2}\right) = \frac{\mathbf{h}(\hat{i})|\mathbf{v}+h_i}{2}$.
(ii) $\dim(\operatorname{aff}(H)) = n$.

These properties imply that $H_{i,\mathbf{v}}$ is a hyperplane of C^{n+1} and therefore $\mathbf{G}_{\mathbf{h}(\hat{i}),\mathbf{v}} \in \tilde{\mathscr{H}}_{n+1}$. Proposition 2.4. says that $\mathbf{G}_{\mathbf{h}(\hat{i}),\mathbf{v}} \in \tilde{\mathscr{H}}_{n+1}$ defines a unique vector $\mathbf{G}(\mathbf{h}, i, \mathbf{v})$ of $\mathscr{H}_n$.

The proof of (2) is immediate since the two last entries of $\mathbf{G}_{\mathbf{h}(\hat{i}),\mathbf{v}}$ and $\mathbf{G}_{\mathbf{h}(\hat{i}),-\mathbf{v}}$ give the same two absolute values. □

Definition 2.4 (*The family* $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$). Using Proposition 2.6 we define recursively for each $n \in \mathbb{N}$ the subset $\mathscr{G}_n \subseteq \mathscr{H}_n$ in the following way:

1. For $n = 1$, $\mathscr{G}_1 := \mathscr{H}_1$.
2. For $n > 1$: $\mathscr{G}_n := \{\mathbf{G}(\mathbf{h}, i, \mathbf{v}) \in \mathscr{H}_{n+1} : \mathbf{h} \in \mathscr{G}_{n-1}, i \in \{1, \ldots, n\}, \mathbf{v} \in C^{n-1}$ such that $\mathbf{h}(\hat{i})|\mathbf{v} \geqslant 0\}$ where for all $\mathbf{h} \in \mathscr{G}_{n-1}$, $i \in \{1, \ldots, n\}$ and $\mathbf{v} \in C^{n-1}$, $\mathbf{G}(\mathbf{h}, i, \mathbf{v})$ denotes the vector of $\mathscr{H}_n$ defined in Proposition 2.6.

The next proposition establishes a bijection between $\mathscr{G}_{n-1}$ and the subset of vectors of $\mathscr{G}_n$ whose first entry is equal to zero.

Proposition 2.7. *For* $n \geqslant 2$, $\mathbf{g} \in \mathscr{G}_{n-1}$ *if and only if* $\mathbf{g}' = (0, \mathbf{g}) \in \mathscr{G}_n$.

Proof. (1) $\mathbf{g} \in \mathscr{G}_{n-1} \Rightarrow \mathbf{g}' = (0, \mathbf{g}) \in \mathscr{G}_n$.

Consider $\mathbf{g} \in \mathscr{G}_{n-1}$ and the hyperplane of C^{n-1} defined by $H : \mathbf{x}|\mathbf{g}(\hat{n}) = g_n$. Let $\mathbf{v}$ be a vertex of $C^{n-1} \cap H$. Then, $\mathbf{G}_{\mathbf{g}(\hat{n}),\mathbf{v}} := \left(\mathbf{g}(\hat{n}), \frac{\mathbf{g}(\hat{n})|\mathbf{v}-g_n}{2}, \frac{\mathbf{g}(\hat{n})|\mathbf{v}+g_n}{2}\right) = (\mathbf{g}(\hat{n}), 0, g_n)$ implying that $\mathbf{G}(\mathbf{g}, n, \mathbf{v}) = (0, \mathbf{g}) \in \mathscr{G}_n$.

(2) $\mathbf{g}' = (0, \mathbf{g}) \in \mathscr{G}_n \Rightarrow \mathbf{g} \in \mathscr{G}_{n-1}$.

We use induction on n to prove this implication.

The case $n = 2$ is trivial. Assume the implication is true for $n < m$ and let $\mathbf{g}' = (0, \mathbf{g})$ be a vector of $\mathscr{G}_m$. Consider $\mathbf{h} \in \mathscr{G}_{m-1}$, $i \in \{1, \ldots, m\}$ and $\mathbf{v} \in C^{m-1}$ such that $\mathbf{v}|\mathbf{h}(\hat{i}) \geqslant 0$ and $\mathbf{g}' = (0, \mathbf{g}) = \mathbf{G}(\mathbf{h}, i, \mathbf{v})$. To simplify the notation we assume that $i = m$, i.e.

$$\mathbf{g}' = (0, \mathbf{g}) = \mathbf{G}(\mathbf{h}, m, \mathbf{v}). \tag{$*$}$$

We consider separately two cases: (A) $\mathbf{h}(\hat{m})$ has all entries greater then zero. (B) $\mathbf{h}(\hat{m})$ has one entry equal to zero.

Case A. In this case by definition of $\mathbf{G}_{\mathbf{h}(\hat{\mathbf{m}}),\mathbf{v}}$ the only entry of this vector which may be zero is the entry $\frac{\mathbf{h}(\hat{m})|\mathbf{v}-h_m}{2}$ implying that $\mathbf{h}(\hat{m})|\mathbf{v} = h_m$ and $\mathbf{g}' = \mathbf{G}(\mathbf{h}, m, \mathbf{v}) = (0, \mathbf{h})$ with $\mathbf{h} \in \mathscr{G}_{m-1}$.

Case B. If the vector $\mathbf{h}(\hat{m})$ has the first entry equal to zero then we write $\mathbf{h}(\hat{m}) = (0, \mathbf{h}(\hat{1}, \hat{m}))$, with $\mathbf{h}(\hat{1}, \hat{m}) \in \mathbb{N}_0^{m-1}$. In this case we have $\mathbf{h} = (0, \mathbf{h}(\hat{1}, \hat{m}), h_m) \in \mathscr{G}_{m-1}$ and the induction assumption implies that $\mathbf{h}(\hat{1}) = (\mathbf{h}(\hat{1}, \hat{m}), h_m)$ is a vector of $\mathscr{G}_{m-2}$.

We claim that $\mathbf{g}$ is the vector $\mathbf{G}(\mathbf{h}(\hat{1}), m, \mathbf{v}(\hat{1}))$ of $\mathscr{G}_{m-1}$, where $\mathbf{v} = (v_1, \mathbf{v}(\hat{1}))$ is a vertex of C^n satisfying ($*$).

Note that $(0, \mathbf{G}_{\mathbf{h}(\hat{1},\hat{m}),\mathbf{v}(\hat{1})}) := \left(0, \mathbf{h}(\hat{1}, \hat{m}), \frac{\mathbf{h}(\hat{1},\hat{m})|\mathbf{v}(\hat{1})-h_m}{2}, \frac{\mathbf{h}(\hat{1},\hat{m})|\mathbf{v}(\hat{1})+h_m}{2}\right) = \left(\mathbf{h}(\hat{m}), \frac{\mathbf{h}(\hat{m})|\mathbf{v}-h_m}{2}, \frac{\mathbf{h}(\hat{m})|\mathbf{v}+h_m}{2}\right) = \mathbf{G}_{\mathbf{h}(\hat{\mathbf{m}}),\mathbf{v}}$. Therefore $(0, \mathbf{G}(\mathbf{h}(\hat{1}), m, \mathbf{v}(\hat{1})) = (0, \mathbf{g})$ and the result follows. $\square$

Remark 2.3. The construction of $\mathscr{G}_n$ from $\mathscr{G}_{n-1}$ is simplified if we consider $\mathscr{G}_n = \mathscr{G}_n^0 \cup \mathscr{G}_n^1$ with $\mathscr{G}_n^0 := \{\mathbf{g} = (g_1, \mathbf{g}(\hat{1})) \in \mathscr{G}_n : g_1 = 0\}$, $\mathscr{G}_n^1 := \{\mathbf{g} = (g_1, \mathbf{g}(\hat{1})) \in \mathscr{G}_n : g_1 \geqslant 1\}$ and determine separately each one of these two sets. Observe that:

(1) By Proposition 2.7. we have $\mathscr{G}_n^0 := \{(0, \mathbf{g}) \in \mathbb{N}_0^{n+1} : \ \mathbf{g} \in \mathscr{G}_{n-1}\}$.
(2) In order to determine $\mathscr{G}_n^1$ we only have to consider those vectors $\mathbf{h} \in \mathscr{G}_{n-1}$ which have at most one entry equal to zero and for those vectors those $i \in \{1, \ldots, n\}$ and $\mathbf{v} \in C^{n-1}$ such that $\mathbf{h}(\hat{i})$ has no entry equal to zero and $\mathbf{v}|\mathbf{h}(\hat{i}) \neq h_i$ ($\mathbf{v}|\mathbf{h}(\hat{i}) \geqslant 0$).

Using the above remark we obtain very easily the explicit description of $\mathscr{G}_n$, for $n \leqslant 6$ given in the next proposition.

Proposition 2.8. *The first* 6 *terms of the sequence* $\mathscr{G}_n$ *are*:

$$\mathscr{G}_1 = \mathscr{H}_1 = \{(1, 1)\};$$
$$\mathscr{G}_2 = \{(0, 1, 1)\};$$
$$\mathscr{G}_3 = \{(0, 0, 1, 1), (1, 1, 1, 1)\};$$
$$\mathscr{G}_4 = \{(0, 0, 0, 1, 1), (0, 1, 1, 1, 1), (1, 1, 1, 1, 2)\};$$

$$\mathscr{G}_5 = \{(0,0,0,0,1,1), (0,0,1,1,1,1), (0,1,1,1,1,2), (1,1,1,1,1,1),$$
$$(1,1,1,1,1,3), (1,1,1,1,2,2), (1,1,1,2,2,3)\};$$
$$\mathscr{G}_6 = \{(0,0,0,0,0,1,1), (0,0,0,1,1,1,1), (0,0,1,1,1,1,2),$$
$$(0,1,1,1,1,1,1), (0,1,1,1,1,1,3), (0,1,1,1,1,2,2),$$
$$(0,1,1,1,2,2,3), (1,1,1,1,1,1,2), (1,1,1,1,1,1,4),$$
$$(1,1,1,1,1,2,3), (1,1,1,1,2,2,2), (1,1,1,1,2,2,4),$$
$$(1,1,1,1,2,3,3), (1,1,1,1,3,3,4), (1,1,1,2,2,2,3),$$
$$(1,1,1,2,2,2,5), (1,1,1,2,2,3,4), (1,1,1,2,3,3,5),$$
$$(1,1,2,2,2,3,3), (1,1,2,2,3,3,4), (1,1,2,2,3,4,5)\}.$$

We have compared the first seven terms of the sequences $\mathscr{G}_n$ and $\mathscr{H}_n$ as well as the numbers g_n and h_n of all the hyperplanes of the n-cube generated by these two sequences. In order to determine $\mathscr{H}_n$ we have developed and used an ad hoc procedure to determine the hyperplanes of C^n. The next theorem summarizes the results obtained.

Theorem 2.1

1. *For* $n \leqslant 6$ *the following equality holds*:

$$\mathscr{G}_n = \mathscr{H}_n.$$

2. *For* $n = 7$ *the following equality holds*:

$$\mathscr{H}_7 = \mathscr{G}_7 \cup \{ (1,1,2,4,4,5,6,7), (1,1,2,4,4,6,7,9), (1,2,3,3,4,4,5,6),$$
$$(1,2,3,4,4,5,6,7), (1,2,3,4,5,6,7,8), (1,3,3,3,4,4,5,5),$$
$$(1,3,3,4,5,5,6,7), (2,2,2,2,3,3,3,5), (2,2,2,3,3,3,4,7),$$
$$(2,2,3,3,3,4,4,5), (2,2,3,3,4,4,5,7), (2,3,3,4,4,5,5,6)\}.$$

Proof. For every $n \in \mathbb{N}$ define $g_n := n^o$ of hyperplanes of C^n generated by $\mathscr{G}_n$ and $h_n := n^o$ of hyperplanes of C^n generated by $\mathscr{H}_n$. In Table 1 we have represented the first seven entries of the integer sequences $|\mathscr{G}_n|$, $|\mathscr{H}_n|$, g_n and h_n.

The entries of the last column h_n are given in [1] and also as sequence A007847 in [7]. The reader may verify the entries of the other three columns in the following way:

Column 1—Determine $\mathscr{G}_n$. The first 6 terms of the sequence $\mathscr{G}_n$ were given in Proposition 2.8 and it is not hard to construct (no machine is needed) the 131 vectors of $\mathscr{G}_7$.

Column 3—Once $\mathscr{G}_n$ is explicitly determined use Remark 2.2. to compute g_n.

Column 2—Since the values of g_n and h_n are equal for $n \leq 6$ the first part of Theorem 2.1 is verified. In order to verify the entry corresponding to $n = 7$ first

Table 1

n	$\|\mathscr{G}_n\|$	$\|\mathscr{H}_n\|$	g_n	h_n [1]
1	1	1	2	2
2	1	1	6	6
3	2	2	20	20
4	3	3	140	140
5	7	7	3254	3254
6	21	21	252,434	252,434
7	131	143	56,541,288	71,343,208

confirm that none of the 12 vectors listed in Theorem 2.1(2)) belongs to $\mathscr{G}_7$. Then, use the definition of $\mathscr{H}_n$ to confirm that these vectors are indeed vectors of $\mathscr{H}_7$. Finally compute (using Remark 2.2) the total number of hyperplanes of C^7 that the vectors of the family $\mathscr{H}_7$ defined in the theorem generate. This number is $h_7 = 71{,}343{,}208$. $\square$

3. Final remarks

In these final remarks we report briefly some further consequences of the results presented.

Consider the Hamming distance in $\mathscr{H}_n$ i.e. the distance defined by:

$$d(\mathbf{h}, \mathbf{h}') := \min_{\sigma \in S_{n+1}} |\{i \in \{1, \ldots, n+1\} : h_{\sigma(i)} \neq h'_i\}|, \quad \forall \mathbf{h}, \mathbf{h}' \in \mathscr{H}_n.$$

Then define the labelled graph of the hyperplanes of C^n—$G(\mathscr{H}_n)$—to be the graph whose vertices are the vectors of $\mathscr{H}_n$ and whose edges are the pairs $\{\mathbf{h}, \mathbf{h}'\}$ such that $1 \leqslant d(\mathbf{h}, \mathbf{h}') \leqslant 2$, labelled by the distance $d(\mathbf{h}, \mathbf{h}')$.

Note that two vertices $\mathbf{h}$, $\mathbf{h}'$ are connected by an edge labelled 1 iff they determine two parallel hyperplanes.

Remark 3.1. By definition of $\mathscr{G}_n$ and induction on n it follows easily that the subgraph $G(\mathscr{G}_n)$ is connected and by Theorem 2.1 we conclude that $G(\mathscr{H}_n)$ is connected for $n \leqslant 6$.

Remark 3.2. $\mathscr{H}_7$ is also connected, as the reader may easily check from the description given in Theorem 2.1 once he computes $\mathscr{G}_7$.

In fact, the twelve vectors of $G(\mathscr{H}_7)$ which are not in $\mathscr{G}_7$ fall into two classes $\mathscr{V}_1$ and $\mathscr{V}_2$ defined as:

$$\begin{aligned}\mathscr{V}_1 &:= \{\mathbf{h} \in \mathscr{H}_7 \setminus \mathscr{G}_7 : \text{ there is a vector } \mathbf{g} \in \mathscr{G}_7 \text{ with } d(\mathbf{g}, \mathbf{h}) = 1\}\\ &= \{(1, 1, 2, 4, 4, 5, 6, 7), (1, 2, 3, 3, 4, 4, 5, 6), (1, 2, 3, 4, 4, 5, 6, 7),\\ &\quad (1, 2, 3, 4, 5, 6, 7, 8), (2, 2, 2, 2, 3, 3, 3, 5), (2, 2, 2, 3, 3, 3, 4, 7),\\ &\quad (2, 2, 3, 3, 3, 4, 4, 5), (2, 2, 3, 3, 4, 4, 5, 7)\}.\end{aligned}$$

$$\begin{aligned}\mathscr{V}_2 &:= \{\mathbf{h} \in \mathscr{H}_7 \setminus \mathscr{G}_7 : \mathbf{h} \notin \mathscr{V}_1 \text{ and there is a vector } \mathbf{g} \in \mathscr{G}_7 \text{ with } d(\mathbf{g}, \mathbf{h}) = 2\}\\ &= \{(1, 1, 2, 4, 4, 6, 7, 9), (1, 3, 3, 3, 4, 4, 5, 5), (1, 3, 3, 4, 5, 5, 6, 7),\\ &\quad (2, 3, 3, 4, 4, 5, 5, 6)\}.\end{aligned}$$

Remark 3.3. A direct consequence of the last remark is that we have a two step procedure to obtain $\mathscr{H}_7$ from $\mathscr{H}_6$:

Step 1. Determine $\mathscr{G}_7$.

Step 2. Look for elements of $\mathscr{H}_7$ of the form $(\mathbf{g}', a, b)$ with $\mathbf{g}'$ restriction of length 6 of a vector of $\mathscr{G}_7$.

Can this procedure be generalized to higher dimensions? In other words, is the graph $G(\mathscr{H}_n)$ always connected? How can Step 2 be made more efficient?

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References

[1] O. Aichholzer, F. Aurenhammer, Classifying hyperplanes in hypercubes, SIAM J. Discrete Math. 8 (1) (1996) 225–232.
[2] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler, Oriented Matroids, Encyclopaedia Math. Appl., vol. 46, Cambridge University Press, 1993.
[3] J. Bokowski, A. Guedes de Oliveira, U. Thiemann, A. Veloso da Costa, On the cube problem of Las Vergnas, Geom. Dedicata 63 (1) (1996) 25–43.
[4] M.R. Garey, D.S. Jonhson, Computers and Intractability, A Guide to the Theory of NP-completeness, W.H. Freeman and Comp., 1979.
[5] M. Hudelson, V. Klee, D. Larman, Largest j-simplices in d-cubes: some relatives of the maximum determinant problem, Linear Algebra Appl. 241–243 (1996) 519–598.
[6] M. Las Vergnas, J.-P. Roudneff, I. Saläun, Regular polytopes and oriented matroids, preprint (1989), 12 pages.
[7] N.J.A. Sloane, On-line Encyclopaedia of integer sequences, http://www.research.att.com/njas/sequences/.